1 Derivation of Eqn (14) in the text

From Equation (7), we have:

$$\beta f_e + \lambda \beta f_{re} = \mu [V'' - (1 - \beta)f_{ee}]$$

This can be written as:

$$\beta f_e [1 + \frac{\lambda}{f_e} f_{re}] = \mu [V'' - (1 - \beta)f_{ee}] \quad (1)$$

From Equation (9), we have (remembering that $f_{er} = f_{re}$):

$$(1 - \beta)f_r + \mu(1 - \beta)f_{re} = \lambda[U'' - \beta f_{rr}]$$

This can be written as:

$$(1 - \beta)f_r [1 + \frac{\mu}{f_r} f_{re}] = \lambda[U'' - \beta f_{rr}] \quad (2)$$

Now, from equation (11) we know that

$$\frac{\lambda}{f_e} = \frac{\mu}{f_r}$$

Therefore, the terms in the square brackets of the LHS in equations are both equal to each other. Then, dividing (1) by (2) gives us:

$$\frac{\beta f_e}{(1 - \beta)f_r} = \frac{\mu}{\lambda} \frac{[V'' - (1 - \beta)f_{ee}]}{[U'' - \beta f_{rr}]}$$

Remembering that $\frac{\mu}{\lambda} = \frac{f_r}{f_e}$, we have:

$$\frac{\beta}{(1 - \beta)} = \frac{f_r^2}{f_e^2} \frac{[V'' - (1 - \beta)f_{ee}]}{[U'' - \beta f_{rr}]} \quad (3)$$

Inverting, adding 1 to both sides, and inverting again, gives us Equation (14) in the text.

2 Derivation of Equation (18) in the text:

Note: there is a typo in Equation (16)

We use (3) and substitute required quantities. Thus,
\[
\begin{align*}
f_r &= \alpha Ke^{\gamma r^\alpha} - 1 \\
&= \frac{\alpha}{r} Ke^{\gamma r^\alpha} \\
\Rightarrow f_r^2 &= \left(\frac{\alpha}{r}\right)^2 (Ke^{\gamma r^\alpha})^2
\end{align*}
\]

Similarly,
\[
\begin{align*}
f_e^2 &= \frac{\gamma}{e^2} (Ke^{\gamma r^\alpha})^2
\end{align*}
\]

Also,
\[
(1 - \beta)f_{ee} = \frac{(1 - \beta)}{e^2} \gamma(\gamma - 1)Ke^{\gamma r^\alpha}
\]

and
\[
\begin{align*}
\beta f_{rr} &= \frac{\beta}{r^2} \alpha(\alpha - 1)Ke^{\gamma r^\alpha}
\end{align*}
\]

Finally,
\[
\begin{align*}
V'' &= \delta E(m - 1)e^{m-2} \\
&= \frac{(m - 1)}{e^2} \delta E e^m \\
&= \frac{(m - 1)}{e^2} \gamma(1 - \beta)Ke^{\gamma r^\alpha}
\end{align*}
\]

(where the last substitution is by using the IC constraint)

and, similarly,
\[
\begin{align*}
U'' = \frac{(n - 1)}{r^2} \alpha \beta Ke^{\gamma r^\alpha}
\end{align*}
\]

Thus,
\[
\begin{align*}
f_r^2 [V'' - (1 - \beta)f_{ee}] &= \left(\frac{\alpha}{r}\right)^2 (Ke^{\gamma r^\alpha})^2 \left[\frac{(m - 1)}{e^2} \gamma(1 - \beta)Ke^{\gamma r^\alpha} - \frac{(1 - \beta)}{e^2} \gamma(\gamma - 1)Ke^{\gamma r^\alpha}\right] \\
&= \frac{\alpha^2}{e^2 r^2} (Ke^{\gamma r^\alpha})^3 \gamma(1 - \beta)[(m - 1) - (\gamma - 1)] \\
&= \frac{\alpha^2}{e^2 r^2} (Ke^{\gamma r^\alpha})^3 \gamma(1 - \beta)(m - \gamma)
\end{align*}
\]

Similarly,
\[
\begin{align*}
f_e^2 [U'' - \beta f_{rr}] &= \frac{\gamma^2}{e^2 r^2} (Ke^{\gamma r^\alpha})^3 \alpha \beta (n - \alpha)
\end{align*}
\]

Dividing one by the other gives:
\[
\begin{align*}
\frac{\beta}{(1 - \beta)} &= \frac{\gamma(1 - \beta)(m - \gamma)}{\alpha \beta (n - \alpha)} \\
\Rightarrow \frac{\beta^2}{(1 - \beta)^2} &= \frac{\gamma(m - \gamma)}{\alpha(n - \alpha)}
\end{align*}
\]

From which Equation (18) follows.