1 Derivation of Eqn (14) in the text

From Equation (7), we have:

$$\beta f_e + \lambda\beta f_{re} = \mu[V'' - (1-\beta)f_{ee}]$$

This can be written as:

$$\beta f_e[1 + \frac{\lambda}{f_e} f_{re}] = \mu[V'' - (1-\beta)f_{ee}] \tag{1}$$

From Equation (9), we have (remembering that $f_{er} = f_{re}$):

$$(1-\beta)f_r + \mu(1-\beta)f_{re} = \lambda[U'' - \beta f_{rr}]$$

This can be written as:

$$(1-\beta)f_r[1 + \frac{\mu}{f_r} f_{re}] = \lambda[U'' - \beta f_{rr}] \tag{2}$$

Now, from equation (11) we know that

$$\frac{\lambda}{f_e} = \frac{\mu}{f_r}$$

Therefore, the terms in the square brackets of the LHS in equations are both equal to each other. Then, dividing (1) by (2) gives us:

$$\frac{\beta f_e}{(1-\beta)f_r} = \frac{\mu}{\lambda} . \frac{[V'' - (1-\beta)f_{ee}]}{[U'' - \beta f_{rr}]}$$

Remembering that$\frac{\mu}{\lambda} = \frac{f_r}{f_e}$, we have:

$$\frac{\beta}{(1-\beta)} = \frac{f_r^2}{f_e^2} . \frac{[V'' - (1-\beta)f_{ee}]}{[U'' - \beta f_{rr}]} \tag{3}$$

Inverting, adding 1 to both sides, and inverting again, gives us Equation (14) in the text.

2 Derivation of Equation (18) in the text:

Note: there is a typo in Equation (16)

We use (3) and substitute required quantities. Thus,

$$\begin{aligned} f_r &= \alpha K e^\gamma r^{\alpha-1} \\ &= \frac{\alpha}{r} K e^\gamma r^\alpha \\ \implies & f_r^2 = (\frac{\alpha}{r})^2 (K e^\gamma r^\alpha)^2 \end{aligned}$$

Similarly,

$$f_e^2 = (\frac{\gamma}{e})^2 (K e^\gamma r^\alpha)^2$$

Also,

$$(1-\beta) f_{ee} = \frac{(1-\beta)}{e^2} \gamma(\gamma-1) K e^\gamma r^\alpha$$

and

$$\beta f_{rr} = \frac{\beta}{r^2} \alpha(\alpha-1) K e^\gamma r^\alpha$$

Finally,

$$\begin{aligned} V'' &= \delta^E (m-1) e^{m-2} \\ &= \frac{(m-1)}{e^2} . \delta^E e^m \\ &= \frac{(m-1)}{e^2} . \gamma(1-\beta) K e^\gamma r^\alpha \end{aligned}$$

(where the last substitution is by using the IC constraint)
and, similarly,

$$U'' = \frac{(n-1)}{r^2} \alpha\beta K e^\gamma r^\alpha$$

Thus,

$$\begin{aligned} f_r^2 . [V'' - (1-\beta) f_{ee}] &= (\frac{\alpha}{r})^2 (K e^\gamma r^\alpha)^2 [\frac{(m-1)}{e^2} . \gamma(1-\beta) K e^\gamma r^\alpha - \frac{(1-\beta)}{e^2} \gamma(\gamma-1) K e^\gamma r^\alpha] \\ &= \frac{\alpha^2}{e^2 r^2} (K e^\gamma r^\alpha)^3 \gamma(1-\beta)[(m-1)-(\gamma-1)] \\ &= \frac{\alpha^2}{e^2 r^2} (K e^\gamma r^\alpha)^3 \gamma(1-\beta)(m-\gamma) \end{aligned}$$

Similarly,

$$f_e^2 . [U'' - \beta f_{rr}] = \frac{\gamma^2}{e^2 r^2} (K e^\gamma r^\alpha)^3 \alpha\beta(n-\alpha)$$

Dividing one by the other gives:

$$\frac{\beta}{(1-\beta)} = \frac{\gamma(1-\beta)(m-\gamma)}{\alpha\beta(n-\alpha)}$$

$$\implies \frac{\beta^2}{(1-\beta)^2} = \frac{\gamma(m-\gamma)}{\alpha(n-\alpha)}$$

From which Equation (18) follows.